HAAGERUP'S APPROXIMATION PROPERTY AND RELATIVE AMENABILITY

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Abstract. A finite von Neumann algebra $\mathcal{M}$ with a faithful normal trace τ has Haagerup's approximation property (relative to a von Neumann subalgebra $\mathcal{N}$) if there exists a net $(\varphi_\alpha)_{\alpha\in\Lambda}$ of normal completely positive ($\mathcal{N}$-bimodular) maps from $\mathcal{M}$ to $\mathcal{M}$ that satisfy the subtracial condition $\tau\circ\varphi_\alpha\leq\tau$, the extension operators T_{φ_α} are bounded compact operators (in $\langle\mathcal{M},e_{\mathcal{N}}\rangle$), and pointwise approximate the identity in the trace-norm, i.e., $\lim_\alpha||\varphi_\alpha(x)-x||_2=0$ for all $x\in\mathcal{M}$. We prove that the subtraciality condition can be removed, and provide a description of Haagerup's approximation property in terms of Connes's theory of correspondences. We show that if $\mathcal{N}\subseteq\mathcal{M}$ is an amenable inclusion of finite von Neumann algebras and $\mathcal{N}$ has Haagerup's approximation property, then $\mathcal{M}$ also has Haagerup's approximation property. This work answers two questions of Sorin Popa.

1. Introduction

A locally compact group G has the *Haagerup property* if there is a sequence of continuous normalized positive definite functions vanishing at infinity on G that converges to 1 uniformly on compact subsets of G. In [Ha1], Haagerup established the seminal result that free groups have the Haagerup property. Now we know that the class of groups having the Haagerup property is quite large. It includes all compact groups, all locally compact amenable groups and all locally compact groups that act properly on trees. There are many equivalent characterizations of the Haagerup property. For instance, G has the Haagerup property if and only if there exists a continuous positive real valued function ψ on G that is conditionally negative definite and proper, i.e., $\lim_{g\to\infty}\psi(g)=0$ (see [AW]). Also the Haagerup property is equivalent to G being a-T-menable in the sense of Gromov (see [Gr1, Gr2, BCV]. An extensive treatment of the Haagerup property for groups can be found in the book [CCJJV]. Studying the class of Haagerup groups has been a fertile endeavor. For example, the Baum-Connes conjecture is solved for this class (see [HK, Tu]).

The Haagerup property is a strong negation to Kazhdan's property T, in that each of the equivalent definitions above stands opposite to a definition of property T (see [CCJJV]). A. Connes and V. Jones introduced a notion of property T for von Neumann algebras in terms of *correspondences* [CJ]. Correspondences, as introduced by Connes (see [Co1, Co2, Po1]), are analogous to group representations in

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the theory of von Neumann algebras. Connes and Jones proved that a group has Kazhdan's property T if and only if the associated group von Neumann algebra has the Connes-Jones property T. Since property T has a natural analogue in the theory of von Neumann algebras, we may expect that the same should be possible for the Haagerup property. Such an analogue indeed exists, as Choda proved in [Ch] that a discrete group has the Haagerup property if and only if its associated group von Neumann algebra has Haagerup's approximation property first introduced in [Ha1]: There exists a net $(\varphi_\alpha)_{\alpha\in\Lambda}$ of normal completely positive maps from $\mathcal{M}$ to $\mathcal{M}$ such that (1) $\tau\circ\varphi_\alpha(x^*x)\leq\tau(x^*x)$ for all $x\in\mathcal{M}$, (2) the extension operator T_{φ_α} of φ_α (see [Po2] or section 2.1 of this paper) is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$ and (3) $\lim_\alpha||\varphi_\alpha(x)-x||_2=0$ for all $x\in\mathcal{M}$. In [Po2], Popa asked if condition (1) can be removed for all finite von Neumann algebras, and proved that if $\mathcal{M}$ is a non-Γ type II_1 factor, then (1) can be removed. In this paper, we prove that (1) can be removed in general.

This enables us to provide a description of Haagerup's approximation property in the language of correspondences. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{H}$ is a correspondence of $\mathcal{M}$. By the Stinespring construction, if φ is a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$, there is a cyclic correspondence $\mathcal{H}_\varphi$ of $\mathcal{M}$ associated to φ. Every correspondence of $\mathcal{M}$ is equivalent to a direct sum of cyclic correspondences associated to completely positive maps as above (see [Po1]). We say that $\mathcal{H}$ is a C_0-*correspondence* if $\mathcal{H}$ is equivalent to $\oplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha}$, where each $\mathcal{H}_{\varphi_\alpha}$ is the correspondence of $\mathcal{M}$ associated to a completely positive map φ_α from $\mathcal{M}$ to $\mathcal{M}$ such that the extension operator T_{φ_α} of φ_α is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$. By the uniqueness of standard representation up to spatial isomorphism(see [Ha2]), the above definition of C_0-correspondence does not depend on the choice of τ. In this paper, we prove that $\mathcal{M}$ has Haagerup's approximation property if and only if the identity correspondence of $\mathcal{M}$ is weakly contained in some C_0-correspondence of $\mathcal{M}$. We also show that if $\mathcal{M}$ has Haagerup's approximation property, then the equivalent class of C_0-correspondences of $\mathcal{M}$ is dense in $\mathrm{Corr}(\mathcal{M})$, the set of equivalent classes of correspondences of $\mathcal{M}$.

In recent breakthrough work, Popa has combined relative versions of property T and Haagerup's approximation property to create "deformation malleability" techniques to solve a number of old open questions about type II_1 factors (see [Po2, IPP]). In [Po2], Popa asked the following question: If $\mathcal{N}\subseteq\mathcal{M}$ is an amenable inclusion of finite von Neumann algebras and $\mathcal{N}$ has Haagerup's approximation property, does $\mathcal{M}$ also have Haagerup's approximation property? Recall that an inclusion $\mathcal{N}\subseteq\mathcal{M}$ of finite von Neumann algebras is amenable if there exists a conditional expectation from the basic construction $\langle\mathcal{M},e_\mathcal{N}\rangle$ onto $\mathcal{M}$ (see [Po1]). This question is motivated by the analogous known result for groups: If G is a subgroup of a discrete group G_0, and G is co-Følner in G_0 in the sense of Eymard [Ey], then whenever G has the Haagerup property so does G_0. A proof of this result can be found in [CCJJV]. The condition that G is co-Følner in G_0 in the sense of Eymard is equivalent to the

amenability of the functorial inclusion $L(G) \subseteq L(G_0)$ of group von Neumann algebras [Po2].

In [Joli], Jolissaint proved that if the basic construction $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ is a finite von Neumann algebra and $\mathcal{N}$ has Haagerup's approximation property, then $\mathcal{M}$ also has Haagerup's approximation property. An affirmative answer to Popa's question for group von Neumann algebras is found in the work of Anantharaman-Delaroche [AD]. Anantharaman-Delaroche proved that the *compact approximation property* is equivalent to the Haagerup approximation property in the group von Neumann algebra case. Recall that a separable finite von Neumann algebra $\mathcal{M}$ has the compact approximation property [AD] if there exists a net $(\phi_\alpha)_{\alpha \in \Lambda}$ of normal completely positive maps from $\mathcal{M}$ to $\mathcal{M}$ such that for all $x \in \mathcal{M}$ we have $\lim_\alpha \phi_\alpha(x) = x$ ultraweakly and for all $\xi \in L^2(\mathcal{M})$ and $\alpha \in \Lambda$, the map $x \mapsto \phi_\alpha(x)\xi$ is a compact operator from *the normed space* $\mathcal{M}$ to $L^2(\mathcal{M})$. Anantharaman-Delaroche proved that if $\mathcal{N} \subseteq \mathcal{M}$ is an amenable inclusion and $\mathcal{N}$ has the compact approximation property, then $\mathcal{M}$ must have the compact approximation property. Using properties (2) and (3) in the above definition of Haagerup's approximation property Anantharaman-Delaroche proved that Haagerup's approximation property implies the compact approximation property. It follows that if $\mathcal{N} \subseteq \mathcal{M}$ is an amenable inclusion and $\mathcal{N}$ has Haagerup's approximation property, then $\mathcal{M}$ also has the compact approximation property. Popa's question is then answered by appealing to the above result of Choda to establish that for group von Neumann algebras the compact approximation property implies Haagerup approximation property.

In this paper, we also answer Popa's second question affirmatively for all finite von Neumann algebras. Our description of Haagerup's approximation property in the language of correspondences plays a key role in the proof. The layout of the rest paper is as follows:

2. Preliminaries
3. Removal of the subtracial condition
4. C_0-correspondences
5. Relative amenability and Haagerup's approximation property

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2. Preliminaries

2.1. **Extension of completely positive maps to Hilbert space operators.** Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\Omega_{\mathcal{M}}$ be the standard trace vector in $L^2(\mathcal{M}, \tau)$ corresponding to $1 \in \mathcal{M}$. For $x, y \in \mathcal{M}$, $\langle x\Omega, y\Omega \rangle_\tau$ is defined to be $\tau(y^*x)$ and $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$. When no confusion arises, we simply write Ω instead of $\Omega_{\mathcal{M}}$, and $\|x\|_2$ instead of $\|x\|_{2,\tau}$.

Suppose φ is a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$. Recall that if there is a $c > 0$ such that $\|\varphi(x)\|_2 \leq c\|x\|_2$ for all $x \in \mathcal{M}$, then there is a (unique) bounded operator T_φ on $L^2(\mathcal{M}, \tau)$ such that

$$T_\varphi(x\Omega) = \varphi(x)\Omega \qquad \forall x \in \mathcal{M}.$$

T_φ is called *the extension operator* of φ. If $\tau \circ \varphi \leq c_0\tau$ for some $c_0 > 0$, then $\|\varphi(x)\|_2 \leq c_0\|\varphi(1)\|^{1/2}\ \|x\|_2$ (see Lemma 1.2.1 of [Po2]) and so there is a bounded operator T_φ on $L^2(\mathcal{M}, \tau)$ such that $T_\varphi(x\Omega) = \varphi(x)\Omega$ for all $x \in \mathcal{M}$.

2.2. **The basic construction and its compact ideal space.** Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\mathcal{N}$ a von Neumann subalgebra of $\mathcal{M}$. The basic construction $\langle \mathcal{M}, e_\mathcal{N}\rangle$ is the von Neumann algebra on $L^2(\mathcal{M}, \tau)$ generated by $\mathcal{M}$ and the orthogonal projection $e_\mathcal{N}$ from $L^2(\mathcal{M}, \tau)$ onto $L^2(\mathcal{N}, \tau)$. Then $\langle \mathcal{M}, e_\mathcal{N}\rangle$ is a semi-finite von Neumann algebra with a faithful normal semi-finite tracial weight Tr such that

$$\mathrm{Tr}(xe_\mathcal{N}y) = \tau(xy), \quad \forall x, y \in \mathcal{M}.$$

Recall that $\langle \mathcal{M}, e_\mathcal{N}\rangle = J\mathcal{N}'J$, where J is the conjugate linear isometry defined by $J(x\Omega) = x^*\Omega$, $\forall x \in \mathcal{M}$. The compact ideal space of $\langle \mathcal{M}, e_\mathcal{N}\rangle$, denoted by $\mathcal{J}(\langle \mathcal{M}, e_\mathcal{N}\rangle)$, is the norm-closed two-sided ideal generated by finite projections of $\langle \mathcal{M}, e_\mathcal{N}\rangle$. Since $e_\mathcal{N}$ is a finite projection in $\langle \mathcal{M}, e_\mathcal{N}\rangle$, it follows that $e_\mathcal{N} \in \mathcal{J}(\langle \mathcal{M}, e_\mathcal{N}\rangle)$. We refer the reader to [Jo, Po2] for more details on the basic construction and its compact ideal space.

2.3. **Correspondences.** Let $\mathcal{N}$ and $\mathcal{M}$ be von Neumann algebras. A correspondence between $\mathcal{N}$ and $\mathcal{M}$ is a Hilbert space $\mathcal{H}$ with a pair of commuting normal representations $\pi_\mathcal{N}$ and $\pi_{\mathcal{M}^\circ}$ of $\mathcal{N}$ and $\mathcal{M}^\circ$ (the opposite algebra of $\mathcal{M}$) on $\mathcal{H}$, respectively. Usually, the triple $(\mathcal{H}, \pi_\mathcal{N}, \pi_{\mathcal{M}^\circ})$ will be denoted by $\mathcal{H}$. For $x \in \mathcal{N}$, $y \in \mathcal{M}$ and $\xi \in \mathcal{H}$, we shall write $x\xi y$ instead of $\pi_\mathcal{N}(x)\pi_{\mathcal{M}^\circ}(y)\xi$. For two vectors $\xi, \eta \in \mathcal{H}$, we denote by $\langle \xi, \eta\rangle_\mathcal{H}$ the inner product of vectors ξ and η. If $\mathcal{N} = \mathcal{M}$, then we simply say $\mathcal{H}$ is a correspondence of $\mathcal{M}$.

Two correspondences $\mathcal{H}, \mathcal{K}$ between $\mathcal{N}$ and $\mathcal{M}$ are *equivalent*, denoted by $\mathcal{H} \cong \mathcal{K}$, if they are unitarily equivalent as $\mathcal{N} - \mathcal{M}$ bimodules (see [Po1]).

2.4. **Correspondences associated to completely positive maps.** Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and φ be a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$. Define on the linear space $\mathcal{H}_0 = \mathcal{M} \otimes \mathcal{M}$ the sesquilinear form

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_\varphi = \tau(\varphi(x_2^*x_1)y_1y_2^*), \quad \forall x_1, x_2, y_1, y_2 \in \mathcal{M}.$$

It is easy to check that the complete positivity of φ is equivalent to the positivity of $\langle \cdot, \cdot\rangle_\varphi$. Let $\mathcal{H}_\varphi$ be the completion of $\mathcal{H}_0/\sim$, where $\sim$ is the equivalence modulo the null space of $\langle \cdot, \cdot\rangle_\varphi$ in $\mathcal{H}_0$. Then $\mathcal{H}_\varphi$ is a correspondence of $\mathcal{M}$ and the bimodule

structure is given by $x(x_1 \otimes y_1)y = xx_1 \otimes y_1 y$ (see [Po1]). We call $\mathcal{H}_\varphi$ the *correspondence of $\mathcal{M}$ associated to φ.*

The correspondence $\mathcal{H}_{\mathrm{id}}$ associated to the identity operator on $\mathcal{M}$ is called the *identity correspondence* of $\mathcal{M}$. It is easy to see that $\mathcal{H}_{\mathrm{id}}$ and $L^2(\mathcal{M}, \tau)$ are equivalent as correspondences of $\mathcal{M}$. The correspondence $\mathcal{H}_{\mathrm{co}}$ associated to the rank one normal completely positive map $\varphi(x) = \tau(x)1$ is called the *coarse correspondence* of $\mathcal{M}$. If $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$ and $E_{\mathcal{N}}$ is the unique τ-preserving normal conditional expectation from $\mathcal{M}$ to $\mathcal{N}$, then the correspondence of $\mathcal{M}$ associated to $E_{\mathcal{N}}$ is denoted by $\mathcal{H}_{\mathcal{N}}$ instead of $\mathcal{H}_{E_{\mathcal{N}}}$.

2.5. **Left τ-bounded vectors.** Let $\mathcal{N}, \mathcal{M}$ be finite von Neumann algebras with faithful normal traces $\tau_{\mathcal{N}}$ and $\tau_{\mathcal{M}}$, respectively, and $\mathcal{H}$ be a correspondence between $\mathcal{N}$ and $\mathcal{M}$. Let $\xi \in \mathcal{H}$ be a vector. Recall that ξ is a *left (or right) τ-bounded vector* if there is a positive number K such that $\langle \xi, \xi x\rangle_{\mathcal{H}} \leq K\tau_{\mathcal{M}}(x)$ (or $\langle x\xi, \xi\rangle_{\mathcal{H}} \leq K\tau_{\mathcal{N}}(x)$, respectively) for all $x \in \mathcal{N}_+$ (or $x \in \mathcal{M}_+$, respectively). A vector ξ is called a *τ-bounded vector* if it is both left τ-bounded and right τ-bounded. The set of τ-bounded vectors is a dense vector subspace of $\mathcal{H}$ (see Lemma 1.2.2 of [Po1]).

2.6. **Coefficients.** Let $\mathcal{N}, \mathcal{M}$ be finite von Neumann algebras with faithful normal traces $\tau_{\mathcal{N}}$ and $\tau_{\mathcal{M}}$, respectively, and $\mathcal{H}$ be a correspondence between $\mathcal{N}$ and $\mathcal{M}$. For a left τ-bounded vector ξ, we can define a bounded operator $T : L^2(\mathcal{M}, \tau_{\mathcal{M}}) \to \mathcal{H}$ by $T(y\Omega_{\mathcal{M}}) = \xi y$ for every $y \in \mathcal{M}$. Let $\Phi_\xi(x) = T^*\pi_{\mathcal{N}}(x)T$, where $\pi_{\mathcal{N}}(x)$ is the left action of $x \in \mathcal{N}$ on $\mathcal{H}$. Then Φ_ξ is a normal completely positive map from $\mathcal{N}$ to $\mathcal{M}$ (see 1.2.1 of [Po1]). Φ_ξ is called the *coefficient* corresponding to ξ, which is uniquely determined by

$$\langle \Phi_\xi(x)y\Omega_{\mathcal{M}}, z\Omega_{\mathcal{M}}\rangle_{\tau_{\mathcal{M}}} = \langle x\xi y, \xi z\rangle_{\mathcal{H}} \tag{1}$$

for all $x \in \mathcal{N}$ and $y, z \in \mathcal{M}$. Therefore,

$$\Phi_\xi(x) = \frac{d\langle x\xi\cdot, \xi\rangle_{\mathcal{H}}}{d\tau_{\mathcal{M}}}, \text{ i.e., } \tau_{\mathcal{M}}(\Phi_\xi(x)y) = \langle x\xi y, \xi\rangle_{\mathcal{H}}, \ \forall x \in \mathcal{N}, y \in \mathcal{M}.$$

If $\mathcal{N} = \mathcal{M}$, $\tau_{\mathcal{N}} = \tau_{\mathcal{M}}$, and $x \geq 0$,

$$\tau_{\mathcal{M}}(\Phi_\xi(x)) = \langle \Phi_\xi(x)\Omega_{\mathcal{M}}, \Omega_{\mathcal{M}}\rangle_\tau = \langle x\xi, \xi\rangle_{\mathcal{H}} \leq K\tau_{\mathcal{M}}(x).$$

By Lemma 1.2.1 of [Po2], Φ_ξ can be extended to a bounded operator T_{Φ_ξ} from $L^2(\mathcal{M}, \tau)$ to $L^2(\mathcal{M}, \tau)$.

It follows by a maximality argument that $\mathcal{H}$ is a direct sum of cyclic correspondences associated to coefficients as above.

2.7. **Composition of correspondences.** Suppose $\mathcal{M}, \mathcal{N}, \mathcal{P}$ are finite von Neumann algebras, and $\tau_{\mathcal{P}}$ is a faithful normal trace on $\mathcal{P}$. Let $\mathcal{H}$ be a correspondence between $\mathcal{N}$ and $\mathcal{P}$ and $\mathcal{K}$ be a correspondence between $\mathcal{P}$ and $\mathcal{M}$. Let $\mathcal{H}'$ and $\mathcal{K}'$ be

vector subspaces of the τ-bounded vectors in $\mathcal{H}$ and $\mathcal{K}$, respectively. For $\xi_1, \xi_2 \in \mathcal{H}'$ and $\eta_1, \eta_2 \in \mathcal{K}'$,

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1 p, \xi_2 \rangle_{\mathcal{H}} = \langle q\eta_1, \eta_2 \rangle_{\mathcal{K}} = \tau_{\mathcal{P}}(qp)$$

defines an inner product on $\mathcal{H}' \otimes \mathcal{K}'$, where p and q are Radon-Nikodym derivatives of normal linear forms $\mathcal{P} \ni z \to \langle z\eta_1, \eta_2 \rangle_{\mathcal{K}}$ and $\mathcal{P} \ni z \to \langle \xi_1 z, \xi_2 \rangle_{\mathcal{H}}$ with respect to the trace $\tau_{\mathcal{P}}$, respectively (see [Po1]). The *composition correspondence* (or the *tensor product correspondence*) $\mathcal{H} \underset{\mathcal{P}}{\otimes} \mathcal{K}$ is the completion of $\mathcal{H}' \otimes \mathcal{K}' / \sim$, where $\sim$ is the equivalence modulo the null space of $\langle \cdot, \cdot \rangle$ in $\mathcal{H}' \otimes \mathcal{K}'$, and the $\mathcal{N} - \mathcal{M}$ bimodule structure is given by $x(\xi \otimes \eta)y = x\xi \otimes \eta y$. It is easy to verify that the composition of correspondences is associative.

2.8. **Induced correspondences.** A very important operation in various representation theories (e.g. for groups) is that of inducing from smaller objects to larger ones. We also have such a concept equally important to the theory of correspondences. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{N}$ a von Neumann subalgebra. If $\mathcal{H}$ is a correspondence of $\mathcal{N}$, then the *induced correspondence* by $\mathcal{H}$ from $\mathcal{N}$ up to $\mathcal{M}$ is $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}) = L^2(\mathcal{M}, \tau) \underset{\mathcal{N}}{\otimes} \mathcal{H} \underset{\mathcal{N}}{\otimes} L^2(\mathcal{M}, \tau)$, where the first $L^2(\mathcal{M})$ is regarded as a left $\mathcal{M}$ and right $\mathcal{N}$ module and the second $L^2(\mathcal{M})$ is regarded as a left $\mathcal{N}$ and right $\mathcal{M}$ module. If $\mathcal{H}$ is the identity correspondence of $\mathcal{N}$, then $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H})$ is the correspondence $\mathcal{H}_{\mathcal{N}}$ of $\mathcal{M}$(see Proposition 1.3.6 of [Po1]).

2.9. **Relative amenability.** Let $\mathcal{H}, \mathcal{K}$ be two correspondences between $\mathcal{N}$ and $\mathcal{M}$. We say that $\mathcal{H}$ is *weakly contained* in $\mathcal{K}$, if for every $\epsilon > 0$, and finite subsets $E \subseteq \mathcal{N}$, $F \subseteq \mathcal{M}$, $\{\xi_1, \cdots, \xi_n\} \subseteq \mathcal{H}$, there exists $\{\eta_1, \cdots, \eta_n\} \subseteq \mathcal{K}$ such that

$$|\langle x\xi_i y, \xi_j \rangle_{\mathcal{H}} - \langle x\eta_i y, \eta_j \rangle_{\mathcal{H}}| < \epsilon,$$

for all $x \in E, y \in F$ and $1 \leq i, j \leq n$. If $\mathcal{H}$ is weakly contained in $\mathcal{K}$, we will denote this by $\mathcal{H} \prec \mathcal{K}$. We refer the reader to [CJ, Po1] for more details on weak containment and topology on correspondences.

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\mathcal{N}$ a von Neumann subalgebra. Recall that $\mathcal{M}$ is *relative amenable to* $\mathcal{N}$ if $\mathcal{H}_{\mathrm{id}} \prec \mathcal{H}_{\mathcal{N}}$. The algebra $\mathcal{M}$ is relative amenable to $\mathcal{N}$ if and only if there exists a conditional expectation from the basic construction $\langle \mathcal{M}, e_{\mathcal{N}} \rangle$ onto $\mathcal{M}$ (see [Po1]).

3. Removal of the Subtracial Condition

The following definition is given by Popa in [Po2].

Definition 3.1. Let $\mathcal{M}$ be a finite von Neumann algebra and $\mathcal{N}$ a von Neumann subalgebra. $\mathcal{M}$ has *Haagerup's approximation property relative to* $\mathcal{N}$ if there exists a normal faithful trace τ on $\mathcal{M}$ and a net of normal completely positive $\mathcal{N}$-bimodular maps $\{\varphi_\alpha\}_{\alpha \in \Lambda}$ from $\mathcal{M}$ to $\mathcal{M}$ satisfying the conditions:

1. $\tau \circ \varphi_\alpha \leq \tau$, $\forall \alpha \in \Lambda$;

2. $T_{\varphi_\alpha} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$, $\forall \alpha \in \Lambda$;
3. $\lim_\alpha \|\varphi_\alpha(x) - x\|_2 = 0$, $\forall x \in \mathcal{M}$.

In [Po2](Remark 2.6), Popa asked if the condition 1 in Definition 3.1 can be removed or not (see Remark 2.6 of [Po1]). In this section we give an affirmative answer to Popa's question. Precisely, we will prove the following theorem.

Theorem 3.2. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{N}$ a von Neumann subalgebra. Suppose $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a net of normal completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying the conditions 2 and 3 as in definition 3.1, i.e. $\lim_\alpha \|\varphi_\alpha(x) - x\|_2 = 0$ for all $x \in \mathcal{M}$ and the map $x\Omega \to \varphi_\alpha(x)\Omega$ extends to a bounded operator T_{φ_α} in $\mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$ for every $\alpha \in \Lambda$. Then there exists a net $\{\psi_\beta\}_{\beta\in\Gamma}$ of normal completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying*

1′. $\psi_\beta(1) = 1$ *and* $\tau \circ \psi_\beta = \tau$, $\forall \beta \in \Gamma$*;*
2′. $T_{\psi_\beta} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$, $\forall \beta \in \Gamma$*;*
3′. $\lim_\beta \|\psi_\beta(x) - x\|_2 = 0$, $\forall x \in \mathcal{M}$.

In particular, $\mathcal{M}$ has Haagerup's approximation property relative to $\mathcal{N}$.

The ideas of the proof of Theorem 3.2 are from Lemma 1.1.1 of [Po2], Day's trick [Da], and Proposition 2.1 of [Joli]. The following lemma is 2 of Lemma 1.1.1 of [Po2] with a minor change. For the sake of completeness, we include the proof.

Lemma 3.3. *Let φ be a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$. Let $a = 1 \vee \varphi(1)$ and $\varphi'(\cdot) = a^{-1/2}\varphi(\cdot)a^{-1/2}$. Then φ' is a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ and satisfies $\varphi'(1) \leq 1$, $\tau \circ \varphi' \leq \tau \circ \varphi$ and the estimate:*

$$\|\varphi'(x) - x\|_2 \leq \|\varphi(x) - x\|_2 + 2\|\varphi(1) - 1\|_2^{1/2} \cdot \|x\|, \quad \forall x \in \mathcal{M}.$$

Proof. Since $a \in \mathcal{N}' \cap \mathcal{M}$, φ' is $\mathcal{N}$-bimodular. We clearly have $\varphi'(1) \leq 1$. Since $a^{-1} \leq 1$, for $x \geq 0$ we get $\tau(\varphi'(x)) \leq \tau(\varphi(x))$. Also, we have:

$$\begin{aligned}\|\varphi'(x) - x\|_2 &\leq \|a^{-1/2}(\varphi(x) - x)a^{-1/2}\|_2 + \|a^{-1/2}xa^{-1/2} - x\|_2 \\ &\leq \|\varphi(x) - x\|_2 + 2\|a^{-1/2} - 1\|_2 \cdot \|x\|.\end{aligned}$$

By the Powers-Størmer inequality (also see Proposition 1.2.1 of [Co3]),

$$\begin{aligned}\|a^{-1/2} - 1\|_2 &\leq \|a^{-1} - 1\|_1^{1/2} = \|a^{-1} - aa^{-1}\|_1^{1/2} \\ &\leq \|a - 1\|_2^{1/2}\|a^{-1}\|_2^{1/2} \leq \|\varphi(1) - 1\|_2^{1/2}.\end{aligned}$$

Thus,

$$\|\varphi'(x) - x\|_2 \leq \|\varphi(x) - x\|_2 + 2\|\varphi(1) - 1\|_2^{1/2} \cdot \|x\|.$$

□

Lemma 3.4. *Let φ be a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ such that $\varphi(1) \leq 1$. Let $b = 1 \vee (d\tau \circ \phi / d\tau)$ and $\varphi'(\cdot) = \varphi(b^{-1/2} \cdot b^{-1/2})$. Then*

φ' is a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ and satisfies $\varphi'(1) \leq \varphi(1) \leq 1$, $\tau \circ \varphi' \leq \tau$ and the estimate:

$$\|\varphi'(x) - x\|_2^2 \leq 2\|\varphi(x) - x\|_2 + 5\|\tau \circ \phi - \tau\|^{1/2} \cdot \|x\|, \quad \forall x \in \mathcal{M}.$$

Proof. Note that $\|\tau \circ \varphi - \tau\| = \|d\tau \circ \varphi/d\tau - 1\|_1$ and $\|b - 1\|_1 \leq \|d\tau \circ \varphi/d\tau - 1\|_1$. Now Lemma 3.4 follows simply from 3 of Lemma 1.1.1 of [Po2]. □

Lemma 3.5. *Let φ be a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ such that $\varphi(1) \leq 1$ and $\tau \circ \varphi \leq \tau$. Let $h = \varphi(1)$ and $k = d\tau \circ \phi/d\tau$. Then $0 \leq h, k \leq 1$, $h, k \in \mathcal{N}' \cap \mathcal{M}$, and $E_{\mathcal{N}}(h) = E_{\mathcal{N}}(k)$.*

Proof. It is easy to see that $0 \leq h, k \leq 1$ and $h, k \in \mathcal{M}$. Since φ is $\mathcal{N}$-bimodular,

$$bh = b\varphi(1) = \varphi(b) = \varphi(1)b = hb, \quad \forall b \in \mathcal{N}.$$

Note that for all $x \in \mathcal{M}$ and $b \in \mathcal{N}$,

$$\begin{aligned}\tau(x(bk - kb)) &= \tau(xbk) - \tau(bxk) = \tau(\phi(xb)) - \tau(\phi(bx)) \\ &= \tau(\phi(x)b) - \tau(b\phi(x)) = 0,\end{aligned}$$

and

$$\begin{aligned}\tau(E_{\mathcal{N}}(h)x) &= \tau(hE_{\mathcal{N}}(x)) = \tau(\varphi(1)E_{\mathcal{N}}(x)) = \tau(\varphi(E_{\mathcal{N}}(x))) \\ &= \tau(E_{\mathcal{N}}(x)k) = \tau(xE_{\mathcal{N}}(k)) = \tau(E_{\mathcal{N}}(k)x).\end{aligned}$$

Hence, $bh = hb$ and $E_{\mathcal{N}}(h) = E_{\mathcal{N}}(k)$. □

Lemma 3.6. *Let φ be a normal completely positive $\mathcal{N}$-bimodular map from $\mathcal{M}$ to $\mathcal{M}$ such that $\varphi(1) \leq 1 - \epsilon$ for some $\epsilon > 0$ and $\tau \circ \varphi \leq \tau$. Let $h = \varphi(1)$ and $k = d\tau \circ \phi/d\tau$. Then there exist positive operators $a, b \in \mathcal{N}' \cap \mathcal{M}$ such that*

$$1 - h = aE_{\mathcal{N}}(b) \quad \textit{and} \quad 1 - k = E_{\mathcal{N}}(a)b.$$

Proof. Let $b = 1 - k$. By Lemma 3.5, b is a positive operator in $\mathcal{N}' \cap \mathcal{M}$ and

$$E_{\mathcal{N}}(b) = 1 - E_{\mathcal{N}}(k) = 1 - E_{\mathcal{N}}(h) = E_{\mathcal{N}}(1 - h).$$

Since $h \leq 1 - \epsilon$, $1 - h \geq \epsilon$ and therefore $E_{\mathcal{N}}(1 - h) \geq \epsilon$. Hence $(E_{\mathcal{N}}(1 - h))^{-1}$ exists. For all $b \in \mathcal{N}$, by Lemma 3.5,

$$bE_{\mathcal{N}}(1 - h) = E_{\mathcal{N}}(b(1 - h)) = E_{\mathcal{N}}((1 - h)b) = E_{\mathcal{N}}(1 - h)b.$$

Hence, $E_{\mathcal{N}}(1-h) \in \mathcal{N} \cap \mathcal{N}'$ and $(E_{\mathcal{N}}(1-h))^{-1} \in \mathcal{N} \cap \mathcal{N}'$. So $a = (1-h)(E_{\mathcal{N}}(1-h))^{-1}$ is a positive operator in $\mathcal{N}' \cap \mathcal{M}$. Since $E_{\mathcal{N}}(b) = E_{\mathcal{N}}(1 - h)$, it is routine to check that $1 - h = aE_{\mathcal{N}}(b)$ and $1 - k = E_{\mathcal{N}}(a)b$. □

Proof of Theorem 3.2. Let $a_\alpha = 1 \vee \varphi_\alpha(1)$ and $\varphi'_\alpha(\cdot) = a_\alpha^{-1/2}\varphi_\alpha(\cdot)a_\alpha^{-1/2}$. By Lemma 3.3, $\{\varphi'_\alpha\}_\alpha$ satisfy the condition $3'$ in Theorem 3.2 and $\varphi'_\alpha(1) \leq 1$ for every $\alpha \in \Lambda$. By Lemma 3.5, $T_{\varphi'_\alpha} = a_\alpha^{-1/2} J a_\alpha^{-1/2} J T_{\varphi_\alpha} \in \langle \mathcal{M}, e_{\mathcal{N}} \rangle$. Since $T_{\varphi_\alpha} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$, $T_{\varphi'_\alpha} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}} \rangle)$ for every $\alpha \in \Lambda$ (see Lemma 1.2.1 of [Po2]).

Let $f_\alpha = \tau \circ \varphi'_\alpha$. Then $\{f_\alpha\}_{\alpha \in \Lambda} \subseteq \mathcal{M}_\#$. Since $\lim_\alpha \|\varphi'_\alpha(x) - x\|_{2,\tau} = 0$ for every x in $\mathcal{M}$, $\lim_\alpha f_\alpha(x) = \tau(x)$ for every $x \in \mathcal{M}$. Since $\mathcal{M}$ is the dual space of $\mathcal{M}_\#$,

this implies that $\lim_\alpha f_\alpha = \tau$ in the weak topology on $\mathcal{M}_\#$. Since the weak closure and the strong closure of a convex set in $\mathcal{M}_\#$ are the same, τ is in the norm closure of the convex hull of $\{f_\alpha\}_{\alpha\in\Lambda}$. Note that $\tau\circ(\sum_{i=1}^n \lambda_{\alpha_i}\varphi'_{\alpha_i}) = \sum_{i=1}^n \lambda_{\alpha_i} f_{\alpha_i}$. By taking finitely many convex combinations of $\{\varphi'_\alpha\}_{\alpha\in\Lambda}$, we can see that there exists a net $\{\psi'_\beta\}_{\beta\in\Gamma}$ of completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying the conditions $2'$ and $3'$ in Theorem 3.2, $\psi'_\beta(1)\le 1$ for all $\beta\in\Gamma$ and the following condition: $\lim_\beta \|g'_\beta - \tau\|_1 = 0$ for $g'_\beta = \tau\circ\psi'_\beta$.

Let $b'_\beta = 1\vee(dg'_\beta/d\tau)$ and $\psi''_\beta(\cdot) = \psi'_\beta((b'_\beta)^{-1/2}\cdot(b'_\beta)^{-1/2})$. By Lemma 3.4, $\{\psi''_\beta\}_{\beta\in\Gamma}$ is a net of completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$, and satisfies $3'$ in Theorem 3.2, $\psi''_\beta(1)\le 1$ and $\tau\circ\psi''_\beta\le\tau$ for all $\beta\in\Gamma$. By Lemma 3.5, $T_{\varphi''_\beta} = T_{\psi'_\beta}{b'_\beta}^{-1/2}J{b'_\beta}^{-1/2}J \in \langle\mathcal{M}, e_\mathcal{N}\rangle$. Since $T_{\psi'_\beta}\in\mathcal{J}(\langle\mathcal{M}, e_\mathcal{N}\rangle)$, $T_{\psi''_\beta}\in\mathcal{J}(\langle\mathcal{M}, e_\mathcal{N}\rangle)$ for every $\beta\in\Gamma$.

We may further assume that $\psi''_\beta(1)\le 1-\epsilon_\beta$, $\epsilon_\beta>0$. Otherwise we can choose a net of positive numbers λ_β with $0<\lambda_\beta<1$ and $\lim_\beta\lambda_\beta = 1$ and consider $\lambda_\beta\cdot\psi''_\beta$. Let $h_\beta = \psi''_\beta(1)$ and $k_\beta = d\tau\circ\psi''_\beta/d\tau$. By Lemma 3.6, there exist positive operators a_β, b_β in $\mathcal{N}'\cap\mathcal{M}$ such that $1-h_\beta = a_\beta E_\mathcal{N}(b_\beta)$ and $1-k_\beta = E_\mathcal{N}(a_\beta)b_\beta$.

For every $\beta\in\Gamma$, define $\psi_\beta:\mathcal{M}\to\mathcal{M}$ by
$$\psi_\beta(x) = \psi''_\beta(x) + a_\beta E_\mathcal{N}(b_\beta x).$$
Clearly, every ψ_β is a normal completely positive $\mathcal{N}$-bimodular map. We have
$$\psi_\beta(1) = \psi''_\beta(1) + a_\beta E_\mathcal{N}(b_\beta) = h_\beta + 1 - h_\beta = 1,$$
and
$$\begin{aligned}\tau(\psi_\beta(x)) &= \tau(\psi''_\beta(x)) + \tau(a_\beta E_\mathcal{N}(b_\beta x)) = \tau(xk_\beta) + \tau(E_\mathcal{N}(a_\beta)b_\beta x)\\ &= \tau(k_\beta x) + \tau((1-k_\beta)x) = \tau(x).\end{aligned}$$
This proves that $\{\psi_\beta\}_{\beta\in\Gamma}$ satisfies the condition $1'$ of Theorem 3.2.

Note that $T_{\psi_\beta} = T_{\psi''_\beta} + a_\beta e_\mathcal{N} b_\beta$. Since $e_\mathcal{N}\in\mathcal{J}(\langle\mathcal{M}, e_\mathcal{N}\rangle)$, $T_{\psi_\beta}\in\mathcal{J}(\langle\mathcal{M}, e_\mathcal{N}\rangle)$ for every $\beta\in\Gamma$. This proves that $\{\psi_\beta\}_{\beta\in\Gamma}$ satisfies the condition $2'$ of Theorem 3.2.

Finally, for every positive operator x in $\mathcal{M}$,
$$\psi_\beta(x) - \psi''_\beta(x) = a_\beta E_\mathcal{N}(b_\beta x) \le \|x\| a_\beta E_\mathcal{N}(b_\beta) = \|x\|(1-h_\beta) = \|x\|(1-\psi''_\beta(1)),$$
which shows that $\{\psi_\beta\}_{\beta\in\Gamma}$ satisfies the condition $3'$ of Theorem 3.2. □

Let τ' be another faithful normal trace on $\mathcal{M}$. Then $\mathcal{M}\subseteq\mathcal{B}(L^2(\mathcal{M},\tau'))$ is in the standard form in the sense of Haagerup [Ha1]. Since the standard representation of a von Neumann algebra is unique up to spatial isomorphism(see [Ha2]), the above arguments indeed prove the following stronger result, which implies that the notion of "relative Haagerup property" considered by Boca in [Bo] is same as Definition 3.1 given by Popa.

Corollary 3.7. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\mathcal{N}$ a von Neumann subalgebra. Suppose $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ is a net of normal completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying the conditions* 2 *and* 3 *as in definition 3.1, i.e.* $\lim_\alpha \|\varphi_\alpha(x) - x\|_{2,\tau} = 0$ *for all* $x \in \mathcal{M}$ *and the map* $x\Omega \to \varphi_\alpha(x)\Omega$ *extends to a bounded operator* T_{φ_α} *in* $\mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}}\rangle)$ *for every* $\alpha \in \Lambda$*. Then for every faithful normal trace* τ' *on* $\mathcal{M}$*, there exists a net* $\{\psi_\beta\}_{\beta\in\Gamma}$ *of normal completely positive $\mathcal{N}$-bimodular maps from $\mathcal{M}$ to $\mathcal{M}$ satisfying*

$1'$ $\psi_\beta(1) = 1$ *and* $\tau' \circ \psi_\beta = \tau'$, $\forall \beta \in \Gamma$*;*

$2'$ $T_{\psi_\beta} \in \mathcal{J}(\langle \mathcal{M}, e_{\mathcal{N}}\rangle)$, $\forall \beta \in \Gamma$*;*

$3'$ $\lim_\beta \|\psi_\beta(x) - x\|_{2,\tau'} = 0$, $\forall x \in \mathcal{M}$.

In particular, the relative Haagerup property does not depend on the choice of faithful normal trace on $\mathcal{M}$.

4. C_0-Correspondences

We now show that Theorem 3.2 enables us to interpret Haagerup's approximation property in the framework of Connes's theory of correspondences. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{H}$ is a correspondence of $\mathcal{M}$.

Definition 4.1. We say that $\mathcal{H}$ is a C_0-*correspondence* if $\mathcal{H} \cong \oplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha}$, where each $\mathcal{H}_{\varphi_\alpha}$ is the correspondence of $\mathcal{M}$ associated to a completely positive map $\varphi_\alpha : \mathcal{M} \to \mathcal{M}$ such that the extension operator T_{φ_α} of φ_α is a compact operator in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.

Remark 4.2. By the uniqueness of standard representation up to spatial isomorphism(see [Ha2]), the definition of C_0-correspondence does not depend on the choice of τ.

Remark 4.3. The coarse correspondence $\mathcal{H}_{\mathrm{co}}$ of $\mathcal{M}$ is a C_0-correspondence. By Proposition 1.2.5 of [Po2], a sub-correspondence of a C_0-correspondence (e.g., the coarse correspondence) is not necessarily a C_0-correspondence. Let $\mathcal{H}_{\mathrm{C}_0}$ be the direct sum of all $\mathcal{H}_\varphi$ such that each $\mathcal{H}_\varphi$ is the correspondence of $\mathcal{M}$ associated to a completely positive map $\varphi : \mathcal{M} \to \mathcal{M}$ with the extension operator T_φ of φ being a compact operator in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. Then $\mathcal{H}_{\mathrm{C}_0}$ is called the *maximal* C_0-*correspondence* of $\mathcal{M}$.

Remark 4.4. Let G be a discrete group, and π be a unitary representation of G on a Hilbert space $\mathcal{H}$. Then π is unitarily equivalent to a direct sum of cyclic representations π_{f_α} of G, where each π_{f_α} is the representation associated to a positive definite function f_α on G. Recall that the representation π is a C_0-representation if all matrix coefficients $\omega_{\xi,\eta}(g) = \langle \pi(g)\xi, \eta\rangle$ belong to $\mathrm{C}_0(G)$. It is easy to check that π is a C_0-representation if and only if every $f_\alpha \in \mathrm{C}_0(G)$. By [Ha1, Ch], for every f_α, there is a unique normal completely positive map φ_{f_α} from the group von Neumann algebra $L(G)$ to itself satisfying $\varphi_{f_\alpha}(L_g) = f_\alpha(g)L_g$, where L_g is the unitary operator associated to g. By Lemma 1 and Lemma 2 of [Ch], f_α is in $\mathrm{C}_0(G)$ if and only if the extension operator $T_{\varphi_{f_\alpha}}$ of φ_{f_α} is a compact operator in $\mathcal{B}(L^2(G))$. Hence, the

correspondence $\mathcal{H}_{\varphi_{f_\alpha}}$ of $L(G)$ associated to φ_{f_α} is a C_0-correspondence of $\mathcal{M}$. So our definition of C_0-correspondence of finite von Neumann algebras is a natural analogue of the notion of C_0-representation of groups.

The following theorem is the main result of this section.

Theorem 4.5. *A finite von Neumann algebra $(\mathcal{M}, \tau)$ has Haagerup's approximation property if and only if the identity correspondence of $\mathcal{M}$ is weakly contained in some C_0-correspondence of $\mathcal{M}$.*

To prove the above theorem, we need the following lemmas.

Lemma 4.6. *Let φ be a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$ such that the extension operator T_φ of φ is a compact operator in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. Let $\xi = \sum_{i=1}^n a_i \otimes b_i$ be a vector in the correspondence $\mathcal{H}_\varphi$ of $\mathcal{M}$ associated to φ. Then ξ is a left τ-bounded vector and the coefficient Φ_ξ corresponding to ξ is a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$ such that T_{Φ_ξ} is a compact operator in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.*

Proof. To see ξ is a left τ-bounded vector, we may assume that $\xi = a \otimes b$. Then

$$\|\xi x\|_\varphi^2 = \langle \xi x, \xi x\rangle_\varphi = \langle a \otimes (bx), a \otimes (bx)\rangle_\varphi = \tau(\varphi(a^*a)bxx^*b^*) = \tau(x^*(b^*\varphi(a^*a)b)x)$$

$$\leq \|b^*\varphi(a^*a)b\|\tau(x^*x) = \|b^*\varphi(a^*a)b\|\|x\|_2^2.$$

Hence Φ_ξ is a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$. For every $x, y, z \in \mathcal{M}$, by equation (1) in section 2.6,

$$\langle \Phi_\xi(x)y\Omega, z\Omega\rangle_\tau = \langle x\xi y, \xi z\rangle_\varphi = \sum_{i,j=1}^n \langle xa_j \otimes b_j y, a_i \otimes b_i z\rangle_\varphi$$

$$= \sum_{i,j=1}^n \tau(\varphi(a_i^* x a_j) b_j y z^* b_i^*) = \langle \sum_{i,j=1}^n b_i^* \varphi(a_i^* x a_j) b_j y\Omega, z\Omega\rangle_\tau.$$

This implies that $\Phi_\xi(x) = \sum_{i,j=1}^n b_i^* \varphi(a_i^* x a_j) b_j$. Hence, Φ_ξ can be extended to a bounded operator from $L^2(\mathcal{M}, \tau)$ to $L^2(\mathcal{M}, \tau)$ such that

$$T_{\Phi_\xi} = \sum_{i,j=1}^n b_i^* J b_j^* J T_\varphi a_i^* J a_j^* J.$$

Since T_φ is a compact operator, T_{Φ_ξ} is also a compact operator. □

Lemma 4.7. *Let $\mathcal{F}$ be the convex hull of the set of coefficients Φ_ξ as in Lemma 4.6. Then $\mathcal{F}$ is a convex cone and for every $b \in \mathcal{M}$ and $\Phi \in \mathcal{F}$, the completely positive map $b^*\Phi(\cdot)b$ belongs to $\mathcal{F}$. Furthermore, T_Φ is a compact operator in $\mathcal{B}(L^2(\mathcal{M}, \tau))$ for all $\Phi \in \mathcal{F}$.*

Proof. It is obvious that $\mathcal{F}$ is a convex cone. To prove the rest, we may assume that $\Phi = \Phi_\xi$ is the coefficient corresponding to $\xi \in \mathcal{H}_\varphi$ as in Lemma 4.6. Let

$\eta = \xi b = \sum_{i=1}^n a_i \otimes b_i b \in \mathcal{H}$. By Lemma 4.6, η is a left τ-bounded vector. Let Φ_η be the coefficient corresponding to η. By equation (1) in section 2.6,

$$\langle \Phi_\eta(x)y\Omega, z\Omega\rangle_\tau = \langle x\xi by, \xi bz\rangle_\varphi = \langle \Phi(x)by\Omega, bz\Omega\rangle_\tau = \langle b^*\Phi(x)by\Omega, z\Omega\rangle_\tau.$$

This implies that $\Phi_\eta = b^*\Phi b$. Hence $b^*\Phi b \in \mathcal{F}$. By Lemma 4.6, T_{Φ_η} is compact. $\square$

Lemma 4.8. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ, and $\mathcal{H}$, $\mathcal{K}$ be two correspondences of $\mathcal{M}$. Suppose $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$ are two left τ-bounded vectors, and Φ_ξ, Φ_η are the coefficients corresponding to ξ, η, respectively. Then $\xi \oplus \eta$ is also a left τ-bounded vector and $\Phi_\xi + \Phi_\eta$ is the coefficient corresponding to $\xi \oplus \eta \in \mathcal{H} \oplus \mathcal{K}$.*

Proof. It is clear that $\xi + \eta$ is a left τ-bounded vector. By equation (1) in section 2.6,

$$\begin{aligned}\langle (\Phi_\xi + \Phi_\eta)(x)y\Omega, z\Omega\rangle_\tau &= \langle x\xi y, \xi z\rangle_{\mathcal{H}} + \langle x\eta y, \eta z\rangle_{\mathcal{K}} = \langle x(\xi \oplus \eta)y, (\xi \oplus \eta)z\rangle_{\mathcal{H}\oplus\mathcal{K}} \\ &= \langle (\Phi_{\xi+\eta}(x)y\Omega, z\Omega\rangle_\tau.\end{aligned}$$

Hence $\Phi_{\xi+\eta} = \Phi_\xi + \Phi_\eta$. $\square$

Note that in the proof of Lemma 2.2 of [AH], if we replace the arbitrary positive normal form ϕ (on line 10 of page 418) by an arbitrary weak operator topology continuous positive form, then the following lemma follows.

Lemma 4.9. *Let Ψ be a normal completely positive map from $\mathcal{M}$ to $\mathcal{M}$. If Ψ is in the closure of $\mathcal{F}$ in the pointwise weak operator topology. Then there exists a net $\{\Phi_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{F}$ such that $\Phi_\alpha(1) \le \Psi(1)$ for all $\alpha \in \Lambda$, which converges to Ψ in the pointwise weak operator topology.*

Proof of Theorem 4.5. Suppose first that $\mathcal{M}$ has Haagerup's approximation property. By Theorem 3.2, there is a net $(\varphi_\alpha)_{\alpha\in\Lambda}$ of unital normal completely positive maps satisfying conditions (1′)-(3′) in Theorem 3.2. It immediately follows that the correspondence $\mathcal{H} = \bigoplus_{\alpha\in\Lambda} \mathcal{H}_{\varphi_\alpha}$ is a C_0-correspondence of $\mathcal{M}$ which weakly contains the identity correspondence of $\mathcal{M}$.

Conversely, suppose that $\mathcal{H}$ is a C_0-correspondence of $\mathcal{M}$ which weakly contains the identity correspondence of $\mathcal{M}$. We may assume $\mathcal{H} = \oplus_{\beta\in\Gamma}\mathcal{H}_{\varphi_\beta}$, with each $\varphi_\beta : \mathcal{M} \to \mathcal{M}$ is a normal completely positive map such that the extension operator T_{φ_β} of φ_β is a compact operator in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. Since the identity correspondence of $\mathcal{M}$ is weakly contained in $\mathcal{H}$, for every $\epsilon > 0$ and every finite subset E of $\mathcal{M}$, there exists a $\xi \in \mathcal{H}$ such that

$$|\langle x\xi y, \xi z\rangle_{\mathcal{H}} - \langle x\Omega y, \Omega z\rangle_\tau| < \epsilon, \quad \forall x, y, z \in E.$$

We may assume that $\xi = \xi_1 \oplus \cdots \oplus \xi_n$, where $\xi_i = \sum_{j=1}^{n_i} a_{ij} \otimes b_{ij} \in \mathcal{H}_{\varphi_{\beta_i}}$. Let Φ_ξ be the coefficient corresponding to ξ. By Lemma 4.7 and Lemma 4.8, $\Phi_\xi \in \mathcal{F}$. By equation (1) in section 2.6,

$$|\langle \Phi_\xi(x)y\Omega, z\Omega\rangle_\tau - \langle x\Omega y, \Omega z\rangle_\tau| = |\langle x\xi y, \xi z\rangle_{\mathcal{H}} - \langle x\Omega y, \Omega z\rangle_\tau| < \epsilon, \quad \forall x, y, z \in E.$$

This implies that there exists a net $(\Phi_{\alpha'})_{\alpha' \in \Lambda'}$ of completely positive maps in $\mathcal{F}$ such that $\lim_{\alpha'} \Phi_{\alpha'}(x) = x$ in the weak operator topology for every $x \in \mathcal{M}$.

By Lemma 4.9, there is a net $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ in $\mathcal{F}$ such that $\lim_\alpha \varphi_\alpha(x) = x$ in the weak operator topology for every $x \in \mathcal{M}$ and $\varphi_\alpha(1) \leq 1$ for every $\alpha \in \Lambda$. Now given $x \in \mathcal{M}$:

$$\begin{aligned} ||\varphi_\alpha(x) - x||_2 &= \tau(\varphi_\alpha(x)^*\varphi_\alpha(x)) + \tau(x^*x) - 2\mathrm{Re}\,\tau(x^*\varphi_\alpha(x)) \\ &\leq ||\varphi_\alpha(1)||\tau(\varphi_\alpha(x^*x)) + \tau(x^*x) - 2\mathrm{Re}\,\tau(x^*\varphi_\alpha(x)) \\ &\leq \tau(\varphi_\alpha(x^*x)) + \tau(x^*x) - 2\mathrm{Re}\,\tau(x^*\varphi_\alpha(x)). \end{aligned}$$

Since $\lim_\alpha \varphi_\alpha(x) = x$ in the weak operator topology for every $x \in \mathcal{M}$ it follows that $\lim_\alpha \tau(\varphi_\alpha(x^*x)) = \tau(x^*x)$ and $\lim_\alpha \tau(x^*\varphi_\alpha(x)) = \tau(x^*x)$. Therefore $\lim_\alpha ||\varphi_\alpha(x) - x||_2 = 0$. This proves that $(\varphi_\alpha)_\alpha$ is a net of completely positive maps that approximate the identity pointwise in the trace-norm. Since $\varphi_\alpha \in \mathcal{F}$, it follows that T_{φ_α} is a compact operator on $L^2(\mathcal{M}, \tau)$. By Theorem 3.2, $\mathcal{M}$ has Haagerup's approximation property. $\square$

As an application of Theorem 4.5, we prove the following theorem.

Theorem 4.10. *If $\mathcal{M}$ has Haagerup's approximation property, then the class of* C_0*-correspondences of $\mathcal{M}$ is dense in* $\mathrm{Corr}(\mathcal{M})$.

Proof. By section 2.6, it is clear that we need only to prove that every cyclic correspondence $\mathcal{H}_\Phi$ of $\mathcal{M}$ associated to a coefficient Φ belongs to the closure of the set of C_0-correspondences of $\mathcal{M}$. Since $\mathcal{M}$ has Haagerup's approximation property, there is a net $\{\varphi_\alpha\}_{\alpha\in\Lambda}$ of normal completely positive maps of $\mathcal{M}$, such that

(1) $\varphi_\alpha(1) = 1$, $\forall \alpha \in \Lambda$,
(2) T_{φ_α} is compact, $\forall \alpha \in \Lambda$,
(3) $\lim_\alpha \|\varphi_\alpha(x) - x\|_2 = 0$, $\forall x \in \mathcal{M}$.

Hence, each $T_{\Phi\circ\varphi_\alpha} = T_\Phi T_{\varphi_\alpha}$ is compact and $\lim_\alpha \|\Phi \circ \varphi_\alpha(x) - \Phi(x)\|_2 = 0$ for every $x \in \mathcal{M}$. By Remark 2.1.4 of [Po1], $\mathcal{H}_{\Phi\circ\varphi_\alpha} \to \mathcal{H}_\Phi$. $\square$

Corollary 4.11. *If $\mathcal{M}$ has Haagerup's approximation property, then every correspondence of $\mathcal{M}$ is weakly contained in $\mathcal{H}_{\mathrm{C}_0}$, the maximal* C_0*-correspondence of $\mathcal{M}$.*

5. Relative Amenability and Haagerup's Approximation Property

Popa asks the following question in [Po2] (see Section 3.5.2): If $\mathcal{N} \subseteq \mathcal{M}$ is an inclusion of finite von Neumann algebras and $\mathcal{N}$ has Haagerup's approximation property, does $\mathcal{M}$ also have Haagerup's approximation property? The following theorem answers Popa's question affirmatively.

Theorem 5.1. *If $\mathcal{N} \subseteq \mathcal{M}$ is an amenable inclusion of finite von Neumann algebras and $\mathcal{N}$ has Haagerup's approximation property then $\mathcal{M}$ also has Haagerup's approximation property.*

To prove Theorem 5.1, we need the following lemmas.

Lemma 5.2. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of finite von Neumann algebras, and $E_{\mathcal{N}}$ be the normal τ-preserving conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. If $\mathcal{H}_\varphi$ is the correspondence of $\mathcal{N}$ associated to a normal completely positive map φ from $\mathcal{N}$ to $\mathcal{N}$ and $\mathcal{H}_{\varphi\circ E_{\mathcal{N}}}$ is the correspondence of $\mathcal{M}$ associated to the normal completely positive map $\varphi \circ E_{\mathcal{N}}$ from $\mathcal{M}$ to $\mathcal{M}$, then $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}_\varphi) \cong \mathcal{H}_{\varphi\circ E_{\mathcal{N}}}$.*

Proof. Denote by $\mathcal{K} = \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}) = L^2(\mathcal{M}) \underset{\mathcal{N}}{\otimes} \mathcal{H}_\varphi \underset{\mathcal{N}}{\otimes} L^2(\mathcal{M})$, where the first $L^2(\mathcal{M})$ is regarded as a left $\mathcal{M}$ and right $\mathcal{N}$ module and the second $L^2(\mathcal{M})$ is regarded as a left $\mathcal{N}$ and right $\mathcal{M}$ module. Let $\xi \in \mathcal{H}_\varphi$ be the vector corresponding to $\Omega \otimes \Omega$, which is a cyclic vector of $\mathcal{H}_\varphi$. Given $x_1, x_2, y_1, y_2 \in \mathcal{M}$, we have

$$\langle (x_1 \otimes (\xi \otimes y_1),\, x_2 \otimes (\xi \otimes y_2)\rangle_{\mathcal{K}} = \langle q(\xi \otimes y_1),\, \xi \otimes y_2\rangle_{\mathcal{H}_\varphi \underset{\mathcal{N}}{\otimes} L^2(\mathcal{M})},$$

where $q \in \mathcal{N}$ is the Radon-Nikodym derivative of $\mathcal{N} \ni z \mapsto \langle x_1 z,\, x_2\rangle_{L^2(\mathcal{M})}$ with respect to $\tau_{\mathcal{N}}$. Note that

$$\langle x_1 z,\, x_2\rangle_{L^2(\mathcal{M})} = \tau(z x_2^* x_1) = \tau(z E_{\mathcal{N}}(x_2^* x_1)).$$

Hence $q = E_{\mathcal{N}}(x_2^* x_1)$ and

$$\langle (x_1 \otimes (\xi \otimes y_1),\, x_2 \otimes (\xi \otimes y_2)\rangle_{\mathcal{K}} = \langle E_{\mathcal{N}}(x_2^* x_1)\xi \otimes y_1,\, \xi \otimes y_2\rangle_{\mathcal{H}_\varphi \underset{\mathcal{N}}{\otimes} L^2(\mathcal{M})} = \langle E_{\mathcal{N}}(x_2^* x_1)\xi p,\, \xi\rangle_{\mathcal{H}_\varphi},$$

where $p \in \mathcal{N}$ is the Radon-Nikodym derivative of $\mathcal{N} \ni z \mapsto \langle z y_1,\, y_2\rangle_{L^2(\mathcal{M})}$ with respect to $\tau_{\mathcal{N}}$. Note that

$$\langle z y_1,\, y_2\rangle_{L^2(\mathcal{M})} = \tau(z y_1 y_2^*) = \tau(z E_{\mathcal{N}}(y_1 y_2^*)).$$

Hence $p = E_{\mathcal{N}}(y_1 y_2^*)$ and

$$\begin{aligned}\langle (x_1 \otimes (\xi \otimes y_1),\, x_2 \otimes (\xi \otimes y_2)\rangle_{\mathcal{K}} &= \langle E_{\mathcal{N}}(x_2^* x_1)\xi p,\, \xi\rangle_{\mathcal{H}_\varphi} = \langle E_{\mathcal{N}}(x_2^* x_1)\xi E_{\mathcal{N}}(y_1 y_2^*),\, \xi\rangle_{\mathcal{H}_\varphi} \\ &= \tau(\varphi(E_{\mathcal{N}}(x_2^* x_1))E_{\mathcal{N}}(y_1 y_2^*)) = \tau(\varphi(E_{\mathcal{N}}(x_2^* x_1)) y_1 y_2^*) = \langle x_1 \xi y_1,\, x_2 \xi y_2\rangle_{\mathcal{H}_{\varphi\circ E_{\mathcal{N}}}}.\end{aligned}$$

Therefore the map defined on simple tensors by $(x_1 \otimes \xi) \otimes x_2 \mapsto x_1 \xi x_2$ extends to an $\mathcal{M}$-linear isometry from $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}_\varphi)$ onto $\mathcal{H}_{\varphi\circ E_{\mathcal{N}}}$. $\square$

The proof of the following lemma is an easy exercise.

Lemma 5.3. *Let $\mathcal{N} \subseteq \mathcal{M}$ be an inclusion of finite von Neumann algebras, and $E_{\mathcal{N}}$ be the normal τ-preserving conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. Suppose for $\alpha \in \Lambda$, $\mathcal{H}_{\varphi_\alpha}$ is the correspondence of $\mathcal{N}$ associated to a normal completely positive map φ_α from $\mathcal{N}$ to $\mathcal{N}$. Then $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\oplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha}) \cong \oplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha\circ E_{\mathcal{N}}}$.*

Lemma 5.4. *If $\mathcal{N} \subseteq \mathcal{M}$ is an inclusion of finite von Neumann algebras and $\mathcal{H}$ is a* C_0*-correspondence of $\mathcal{N}$, then $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H})$ is a* C_0*-correspondence of $\mathcal{M}$.*

Proof. Let $E_{\mathcal{N}}$ be the τ-preserving normal conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$. Suppose $\mathcal{H} = \bigoplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha}$ such that T_{φ_α} is a compact operator in $\mathcal{B}(L^2(\mathcal{N},\tau))$. By Lemma 5.2 and Lemma 5.3 we have that $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\bigoplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha}) \cong \bigoplus_{\alpha\in\Lambda}\mathcal{H}_{\varphi_\alpha\circ E_{\mathcal{N}}}$. Since $T_{\varphi_\alpha\circ E_{\mathcal{N}}} = T_{\varphi_\alpha} e_{\mathcal{N}}$, the operator $T_{\varphi_\alpha\circ E_{\mathcal{N}}}$ is a compact operator in $\mathcal{B}(L^2(\mathcal{M},\tau))$. So $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H})$ is a C_0-correspondence of $\mathcal{M}$. $\square$

Proof of Theorem 5.1. Let $\mathcal{H}$ be a C$_0$-correspondence of $\mathcal{N}$ that weakly contains the identity correspondence $L^2(\mathcal{N},\tau)$ of $\mathcal{N}$. By Lemma 5.4, $\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H})$ is a C$_0$-correspondence of $\mathcal{M}$. Note that $L^2(\mathcal{N},\tau) \prec \mathcal{H}$. By the continuity of induction operation (see Proposition 2.2.1 of [Po1]), we see that

$$\mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(L^2(\mathcal{N},\tau)) \prec \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}).$$

Since $\mathcal{N} \subseteq \mathcal{M}$ is an amenable inclusion, we have

$$L^2(\mathcal{M},\tau) \prec \mathcal{H}_{\mathcal{N}} = \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(L^2(\mathcal{N},\tau)).$$

By the transitivity of $\prec$ we obtain

$$L^2(\mathcal{M},\tau) \prec \mathrm{Ind}_{\mathcal{N}}^{\mathcal{M}}(\mathcal{H}).$$

By Theorem 4.5, $\mathcal{M}$ has Haagerup's approximation property. □

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